Lecture 6

Wednesday, 9 September 2026 5:54 PM

Last class:

- A **simplex** is the convex hull of n affinely independent vectors

  i.e. $\sigma = \left\{ \sum_{i=1}^{n} \lambda_i v_i ; \ \sum_{i=1}^{n} \lambda_i = 1, \ \forall i \ \lambda_i \geq 0 \right\}$

  $v_1, \ldots, v_n$ are affinely independent

Triangulation (of σ): A collection of simplices of dimension $(n-1)$, so that:
① their union is σ,
② any two simplices in the collection either have no intersection, or intersect in a common face of both.
These are called **elementary simplices**.

Consider the simplex Δ_{n-1}, triangulation T (with elementary simplices σ).
Let $V(T)$ be vertices of all the simplices.
Then $l: V(T) \to [n]$ is a Sperner labelling if
$l(x) \in \{ i : x_i > 0 \}$

An elementary simplex is called a rainbow simplex if its vertices have all n colours.

Sperner's lemma: There are an odd number of rainbow simplices (thus, at least 1).

Given a cake $C = [0,1]$, a cut-set $x = (x_1, \ldots, x_n)$ s.t. $\forall i$, $x_i \geq 0$, and $\sum_i x_i = 1$.
Clearly, pts. in Δ_{n-1} correspond exactly to cut-sets.
Each cut-set also gives a connected cake allocation, where
$A_1 = [0, x_1]$, $A_2 = [x_1, x_1 + x_2]$, $A_3 = [x_1 + x_2, x_1 + x_2 + x_3]$, ...
We refer to A_i as the i^{th} piece for a cut-set x.

Given a cut-set x, agent i "prefers" piece k if $A_k \in \arg\max_{j \in [n]} v_i(A_j)$
(thus A_k has maximum value for agent i)

Lemma: Let f_i be the valuation density fn. for agent i, & $f_i(x) > 0$ $\forall i \in N$, $x \in C$. Then if (x^k) is a sequence of cutsets converging to $\hat{x}$, & if agent i prefers piece j in each of $x^1, \ldots, x^k$, then i prefers piece j in $\hat{x}$ as well
(prove yourself)

(we say the sequence $x^k = x^1, x^2, \ldots$ converges to x^* if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall k \geq N$, $\|x^k - x^*\|_2 \leq \varepsilon$)

Theorem: $\exists$ a connected EF allocation.

Barycentric subdivision: For the standard simplex Δ_{n-1}, for every $S \subseteq [n]$, consider the face σ_S. The **barycentre** of σ_S is the point:

$$b_{S,i} = \begin{cases} \frac{1}{|S|} & \text{for } i \in S \\ 0 & \text{o.w.} \end{cases}$$

Thus for every $S \subseteq [n]$, there is a barycentre b_S.
Now consider the lattice consisting of subsets of $[n]$
For every maximal chain in this lattice
$\phi \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq S_n = [n]$
we construct an elementary simplex w/ vertices $\{ b_\phi, b_{S_1}, b_{S_2}, \ldots, b_{S_n} \}$
(there are thus $n!$ elementary simplices)
This is called the barycentric subdivision of Δ_{n-1}
Claim: This is a triangulation of Δ_{n-1}
(check yourself)

For a simplex σ, define diameter $(\sigma) = \max_{x,y \in \sigma} \|x - y\|_2$

Claim: Let T be a barycentric subdivision of σ, & $\hat{\sigma}$ be an elementary simplex. Then $\operatorname{diam}(\hat{\sigma}) \leq \left(1 - \frac{1}{n+1}\right) \operatorname{diam}(\sigma)$
(prove yourself)

Barycentric labelling: In a barycentric subdivision, vertices are b_S, $S \subseteq [n]$.
$\beta(b_S) = |\operatorname{supp}(b_S)| + 1 \in [n]$

Note that in a barycentric subdivision of Δ_{n-1}, every elementary simplex has a different β-label on each vertex.
We say agent i **owns** a vertex, if $\beta(v) = i$.

Now let $T^{(1)}$ be a Barycentric triangulation
$T^{(2)}, T^{(3)}, \ldots$ be a sequence of triangulations, where $T^{(i+1)}$ is obtained by a barycentric triangulation of the elementary simplices in $T^{(i)}$.
& $\beta: V(T^{(i)}) \to [n]$ is a labelling of $V(T^{(i)})$ so that for each elementary simplex in $T^{(i)}$, each vertex gets a different label
(can show by induction this exists)

Proof: Given Δ_{n-1}, let T be a barycentric subdivision. Then recall that every elementary simplex has a different β-label on each vertex; $\beta(x) \in [n]$. (or equivalently, each agent i owns a distinct vertex v_i)

Now we construct a different labeling on the vertices $V(T)$.
For each vertex $x \in V(T)$, let $l(x) \in [n]$ be a piece of cake preferred by agent $\beta(x)$, breaking ties arbitrarily.
Thus at a simplex, we ask a different agent which piece they prefer at each vertex.
Claim: $l: V(T) \to [n]$ is a Sperner labelling.
Proof: We need to show that for each $x \in V(T)$, $l(x) \in \{ i : x_i > 0 \}$.
But this is immediate, since agents are "hungry".
Thus, $\exists$ a rainbow simplex: an elementary simplex where each vertex is a different colour.
(Equivalently, an elementary simplex where each agent prefers a different piece of cake)
Now, let $\sigma^{(1)}, \sigma^{(2)}, \ldots$ be a rainbow simplex in $T^{(1)}, T^{(2)}, \ldots$
(& note that $\sigma^{(k)}$'s are getting smaller & smaller, i.e., diameter $(\sigma^{(k)}) \to 0$ as $k \to \infty$)
Each $\sigma^{(k)}$ satisfies:
- each vertex has a different β-label (by Barycentric labelling)
  (i.e., each agent owns a diff. vertex)
- each vertex has a different Sperner label
  (i.e., each agent prefers a different piece of cake)

Now for each $\sigma^{(k)}$, let $v_i^{(k)}$ be the vertex of $\sigma^{(k)}$ w/ β-label i
(this is the vertex owned by agent i).
Further let $b^{(k)}$ be the barycentre of simplex $\sigma^{(k)}$.
Let us assume the following, which we will justify later.
① $\forall i \in N$, $\exists n_i \in [n]$ s.t. agent i prefers the n_i^{th} piece at each vertex $\{ v_i^{(k)} \}$
Note that the labels n_i form a Sperner colouring for each simplex $\{ \sigma^{(k)} \}$, and hence $n_i \neq n_j$ for agents $i \neq j$
(thus for each simplex $\{ \sigma^{(k)} \}$, each agent i prefers a diff piece of cake.
② The sequence $b^{(1)}, \ldots$ is convergent, & converges to x^*.

Proof of theorem: Firstly note that $\forall i$, $\{ v_i^{(k)} \} \to x^*$, since diameter $(\sigma^{(k)})$ decreases geometrically & $b^{(k)} \to x^*$.
Also for each agent i, she prefers the n_i^{th} piece at each vertex $v_i^{(k)}$, and hence also at x^*. But for each simplex $\sigma^{(k)}$, the n_i's form a Sperner labelling, & each $\sigma^{(k)}$ is a rainbow simplex. Hence $n_i \neq n_j$ for agents $i \neq j$, & x^* is an EF & connected partition of the cake. ∎

Assumption 1:
Since $\{ \sigma^{(k)} \}$ is an infinite sequence, & at each rainbow simplex there are $n!$ possible preferences for the agents, we simply pick an infinite subsequence where $\forall i$, $\exists$ a piece n_i s.t. agent i prefers piece n_i at each simplex in the subsequence.

Assumption 2:
Now, let $\{ \sigma^{(k)} \}$ be a sequence where $\forall i$, $\exists n_i \in [n]$ s.t. agent i prefers piece n_i in $\{ \sigma^{(k)} \}$.
Want to show $\exists x^* \in \Delta_{n-1}$ s.t. $b^{(k)} \to x^*$ as $k \to \infty$
... but this is not strictly true.
Bolzano – Weierstrass Theorem: Every sequence in a closed and bounded set has a convergent subsequence
(w/o proof)
Thus, the sequence $\{ b^{(k)} \}$ has a convergent subsequence.
That is, $\exists$ an infinite subset $S' \subseteq \mathbb{Z}_+$ s.t. the subsequence $(b^{(k)})_{k \in S'}$ converges.
Instead of the original sequence, we just consider this subsequence instead.